

A violation of the uncertainty principle implies a violation of the second law of thermodynamics

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Uncertainty relations state that there exist certain incompatible measurements, to which the outcomes cannot be simultaneously predicted. While the exact incompatibility of quantum measurements dictated by such uncertainty relations can be inferred from the mathematical formalism of quantum theory, the question remains whether there is any more fundamental reason for the uncertainty relations to have this exact form. What, if any, would be the operational consequences if we were able to go beyond any of these uncertainty relations? We give a strong argument that justifies uncertainty relations in quantum theory by showing that violating them implies that it is also possible to violate the second law of thermodynamics. More precisely, we show that violating the uncertainty relations in quantum mechanics leads to a thermodynamic cycle with positive net work gain, which is very unlikely to exist in nature.

Many features commonly associated with quantum physics, such as the uncertainty principle [1] or non-locality [2] appear highly counter-intuitive at first sight. The fact that quantum mechanics is more non-local than any classical theory [2], but yet more limited [3, 4] than what the no-signalling principle alone demands [5–7] has been the subject of much investigation [8–12]. Several reasons and principles were put forward that explain the origin of such quantum mechanical limits [10–12].

In [12] it was shown that the amount of non-locality in quantum mechanics is indeed directly related to another fundamental quantum mechanical limit, namely the uncertainty principle [1]. This forged a relation between two fundamental quantum mechanical concepts. We may however still ask why the uncertainty principle itself is not maybe stronger or weaker than predicted by quantum physics? - and, what would happen if it was?

Here we relate this question to the second law of thermodynamics. We show that any violation of uncertainty relations in quantum mechanics also leads to a violation of the second law.

BACKGROUND

To state our result, we need to explain three different concepts. First, we need some properties of generalized physical theories (see e.g. [13–17]). Second, we recall the concept of uncertainty relations, and finally the second law of thermodynamics.

Physical theories Whereas it is not hard to prove our result for quantum theory, we extend our result to some more general physical theories. These are described by a probabilistic framework that makes the minimal assumptions that there are *states* and *measurements* which

can be made on a physical system (see, e.g., [18, 19]). Even for general theories, we denote a state as $\rho \in \Omega$, where Ω is a convex state space. In quantum mechanics, ρ is simply a density matrix. The assumption that the state space is convex is thereby generally made [17] and says that if we can prepare states ρ_1 and ρ_2 , then the probabilistic mixture $\rho = \rho_1/2 + \rho_2/2$ prepared by tossing a coin and preparing ρ_1 or ρ_2 with probability $1/2$ each is also an element of Ω . A state is called *pure* if it cannot be written as a convex combination of other states. Measurements consist of linear functionals $e_j : \Omega \rightarrow [0, 1]$ called *effects*. We call an effect e_j *pure* if it cannot be written as a positive linear combination of any other allowed effects. Intuitively, each effect corresponds to a possible measurement outcome, where $p(e_j|\rho) = e_j(\rho)$ is the probability of obtaining "outcome" e_j given the state ρ . More precisely, a measurement is thus given by $\mathbf{e} = \{e_j \mid \sum_j p(e_j|\rho) = 1\}$. For quantum mechanics, we will simply label effects by measurement operators. For example, a projective measurement in the eigenbasis $\{0_Z, 1_Z\}$ of the Pauli Z operator is denoted by $p(0_Z|\rho) = \text{tr}(|0_Z\rangle\langle 0_Z|\rho)$. The assumption that effects are linear, i.e., $p(e_j|\rho)$ is linear in ρ , is essentially made for all probabilistic theories [17] and says that when we prepared a probabilistic mixture of states the distribution of measurement outcomes scales accordingly.

Uncertainty relations A modern way of quantifying uncertainty [20, 21] is by means of *entropic uncertainty relations* (see [22] for a survey), or the closely related *fine-grained uncertainty relations* [12]. Here we will use the latter. As for our cycle we will only need two measurements with two outcomes, and each measurement is chosen with probability $1/2$. We state their definition only for this simple case. Let $\mathbf{f} = \{f_0, f_1\}$ and $\mathbf{g} = \{g_0, g_1\}$ denote the two measurements with effects f_{y_1} and g_{y_2} respectively. A fine-grained uncertainty relation for these measurements is a set of inequalities

$$\left\{ \forall \rho : \frac{1}{2} (p(f_{y_1}|\rho) + p(g_{y_2}|\rho)) \leq \zeta_{\vec{y}} \mid \vec{y} \in \{0, 1\}^2 \right\}. \quad (1)$$

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To see why this quantifies uncertainty, note that if $\zeta_{\vec{y}} < 1$ for some $\vec{y} = (y_1, y_2)$, then we have that if the outcome is certain for one of the measurements (e.g., $p(f_{y_1}|\rho) = 1$) it is uncertain ($p(g_{y_2}|\rho) < 1$) for the other. As an example from quantum mechanics, consider measurements in the $X = \{0_X, 1_X\}$ and $Z = \{0_Z, 1_Z\}$ eigenbases.¹ We then have for all pure quantum states ρ

$$\frac{1}{2} (p(0_X|\rho) + p(0_Z|\rho)) \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}. \quad (2)$$

The same relation holds for all other pairs of outcomes $(0_X, 1_Z), (1_X, 0_Z)$ and $(1_X, 1_Z)$. Depending on \vec{y} , the eigenstates of either $(X+Z)/\sqrt{2}$ or $(X-Z)/\sqrt{2}$ saturate these inequalities. A state that saturates a particular inequality is also called a *maximally certain state* [12].

For any theory such as quantum mechanics in which there is a direct correspondence between *states* and *measurements* uncertainty relations can also be stated in terms of *states* instead of measurements. More precisely, uncertainty relations can be written in terms of states if pure effects and pure states are dual to each other in the sense that for any pure effect f there exists a corresponding pure state ρ_f , and conversely for every pure state σ an effect e_σ such that $p(f|\sigma) = p(e_\sigma|\rho_f)$. Here, we restrict ourselves to theories that exhibit such a duality. This is often (but not always) assumed [17, 19]. As a quantum mechanical example, consider the effect $f = 0_X$ and the state $\sigma = |0\rangle\langle 0|$. We then have $p(f|\sigma) = \text{tr}(|+\rangle\langle +|\sigma) = \text{tr}(|+\rangle\langle +||0\rangle\langle 0|) = p(e_\sigma|\rho_f)$ with $\rho_f = |+\rangle\langle +|$ and $e_\sigma = 0_Z$.

For measurements $\mathbf{f} = \{f_0, f_1\}$ and $\mathbf{g} = \{g_0, g_1\}$ consisting of pure effects, let $\{\rho_{f_0}, \rho_{f_1}\}$ and $\{\rho_{g_0}, \rho_{g_1}\}$ denote the corresponding dual states. The equations of (1) then take the dual form

$$\forall \text{ pure effects } e : \frac{1}{2} (p(e|\rho_{f_{y_1}}) + p(e|\rho_{g_{y_2}})) \leq \zeta_{\vec{y}}. \quad (3)$$

For our quantum example of measuring in the X and Z eigenbasis, we have $\rho_{0_X} = |+\rangle\langle +|$, $\rho_{1_X} = |-\rangle\langle -|$, $\rho_{0_Z} = |0\rangle\langle 0|$ and $\rho_{1_Z} = |1\rangle\langle 1|$. We then have that for all pure quantum effects e

$$\frac{1}{2} (p(e|\rho_{0_X}) + p(e|\rho_{1_Z})) \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}. \quad (4)$$

The same relation holds for all other pairs $(0_X, 1_Z), (1_X, 0_Z)$ and $(1_X, 1_Z)$. Again, measurement effects from the eigenstates of either $(X+Z)/\sqrt{2}$ or $(X-Z)/\sqrt{2}$ saturate these inequalities. In analogy, with maximally certain states we refer to effects that saturate the inequalities (3) as *maximally certain effects*. From

now on, we will always consider uncertainty relations in terms of *states*.

2nd law The second law of thermodynamics is usually stated in terms of entropies. One way to state it is to say that the entropy of an isolated system cannot decrease. These entropies can be defined for general physical theories even for systems which are not described by the quantum formalism [19, 23, 24] (see appendix). However, for our case it will be sufficient to consider one operational consequence of the second law of thermodynamics [25, 26]: there cannot exist a cyclic physical process with a net work gain over the cycle.

RESULT

Our main result is that if it was possible to violate the fine-grained uncertainty relations as predicted by quantum physics, then we could create a cycle with net work gain. This holds for *any* two projective measurements with two outcomes on a qubit. By the results of [12] which showed that the amount of non-locality is solely determined by the uncertainty relations of quantum mechanics and our ability to steer, our result extends to a link between the amount of non-locality and the second law of thermodynamics.

In the following we focus on the quantum case, i.e., in the situation where all the properties except the uncertainty relations hold as for quantum theory. In the appendix, we extend our result to more general physical theories that satisfy certain assumptions. In essence, different forms of entropies coincide in quantum mechanics, but can differ in more general theories [19]. This has consequences on whether a net work gain in our cycle is due to a violation of uncertainty alone, or can also be understood as the closely related question of whether certain entropies can differ.

Let us now first state our result for quantum mechanics more precisely. We consider the following process as depicted in Figure 1. We start with a box which contains two types of particles described by states ρ_0 and ρ_1 in two separated volumes. The state ρ_0 is the equal mixture of the eigenstates ρ_{f_0} and ρ_{g_0} of two measurements (observables) $\mathbf{f} = \{f_0, f_1\}$ and $\mathbf{g} = \{g_0, g_1\}$. The state ρ_1 is the equal mixture of ρ_{f_1} and ρ_{g_1} . We choose the measurements such that the equal mixture $\rho = (\rho_0 + \rho_1)/2$ is the completely mixed state in dimension 2. We then replace the wall separating ρ_0 from ρ_1 by two semi-transparent membranes, i.e., membranes which measure any arriving particle in a certain basis $\mathbf{e} = \{e_0, e_1\}$ and only let it pass for a certain outcome. In the first part of the cycle we separate the two membranes until they are in equilibrium, which happens when the state everywhere in the box can be described as ρ . Then, in the second part of the cycle, we separate ρ again into its different components.

We find that the total work which can be extracted by

¹ We use the common convention of labelling the X and Z eigenbases states as $\{|+\rangle, |-\rangle\}$ and $\{|0\rangle, |1\rangle\}$ respectively.

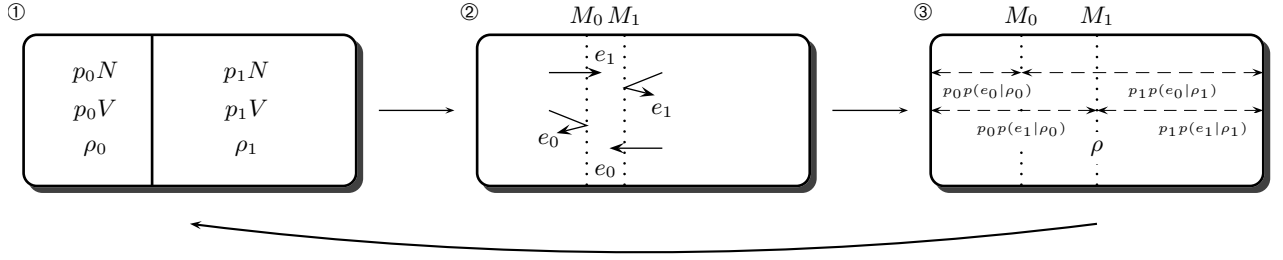


FIG. 1: The impossible process.

performing this cycle is given by

$$\Delta W = NkT \ln 2 \left(\sum_{i=0}^1 p_i S(\rho_i) - \frac{1}{2} H(\zeta_{(f_0, g_0)}) - \frac{1}{2} H(\zeta_{(f_1, g_1)}) \right).$$

Here, $S(\rho) = -\text{tr}(\rho \log \rho)$ is the *von Neumann entropy* of the state. The entropy H appearing in the above expression is simply the Shannon entropy of the distribution over measurement outcomes when measuring in the basis \mathbf{f} and \mathbf{g} , respectively.²

Example To illustrate our result, consider the concrete quantum example, where the states are given by

$$\begin{aligned} \rho_0 &= \frac{1}{2}(\rho_{0x} + \rho_{0z}) = \frac{\mathbb{1} + \frac{X+Z}{2}}{2} \text{ and} \\ \rho_1 &= \frac{1}{2}(\rho_{1x} + \rho_{1z}) = \frac{\mathbb{1} - \frac{X+Z}{2}}{2}. \end{aligned} \quad (5)$$

The work which can be extracted from the cycle then becomes

$$\Delta W = NkT \ln 2 \left(H\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right) - \frac{1}{2} H(\zeta_{(0x, 0z)}) - \frac{1}{2} H(\zeta_{(1x, 1z)}) \right).$$

The fine-grained uncertainty relations predict in the quantum case that $\zeta_{(0x, 0z)}$ and $\zeta_{(1x, 1z)}$ are at most $\frac{1}{2} + \frac{1}{2\sqrt{2}}$. We see that a theory which can violate this uncertainty relation, i.e., reach a larger value of ζ , would lead to $\Delta W > 0$ — a violation of the second law of thermodynamics.

METHODS

We now explain in more detail how we obtain the work which can be extracted from the cycle in quantum me-

chanics. In the appendix, we consider the case of general physical theories.

First part of the cycle

For the first part of the cycle we start with two separate parts of the box in each of which there are $N/2$ particles in the states ρ_0 and ρ_1 respectively. These states are described by

$$\begin{aligned} \rho_0 &= \frac{1}{2}(\rho_{f_0} + \rho_{g_0}) \text{ and} \\ \rho_1 &= \frac{1}{2}(\rho_{f_1} + \rho_{g_1}), \end{aligned}$$

where $\mathbf{f} = \{f_0, f_1\}$ and $\mathbf{g} = \{g_0, g_1\}$ are chosen such that the state $\rho = \rho_0/2 + \rho_1/2$ corresponds to the completely mixed state in dimension 2. We then make a projective measurement $\mathbf{e} = \{e_0, e_1\}$ with two possible outcomes denoted by 0, 1. More precisely, we insert two semi-transparent membranes instead of the wall separating the two volumes. One of the membranes is transparent to e_0 but completely opaque to e_1 while the other lets the particle pass if the outcome is e_1 , but not if it was e_0 . Letting these membranes move apart until they are in equilibrium, we can extract work from the system. The equilibrium is reached when on both sides of the membranes which is opaque for e_1 , there is the same density of particles in this state and similarly for the membrane which is opaque for e_0 .

The work which can be extracted from the first part of the cycle (i.e., by going from ① to ③ in Figure 1) is given by the following (see appendix).

$$\begin{aligned} W &= NkT \ln 2 \left(1 - \frac{1}{2} H\left(\frac{1}{2} p(e_0|\rho_{f_0}) + \frac{1}{2} p(e_0|\rho_{g_0})\right) \right. \\ &\quad \left. - \frac{1}{2} H\left(\frac{1}{2} p(e_1|\rho_{f_1}) + \frac{1}{2} p(e_1|\rho_{g_1})\right) \right) \\ &\leq NkT \ln 2 \left(1 - \frac{1}{2} H(\zeta_{(f_0, g_0)}) - \frac{1}{2} H(\zeta_{(f_1, g_1)}) \right), \end{aligned}$$

where we denoted by ζ the fine-grained uncertainty relations. The inequality can be saturated by choosing e_0 and

² The Shannon entropy of a probability distribution $\{p_1, \dots, p_d\}$ is given by $H(\{p_1, \dots, p_d\}) = -\sum_j p_j \log p_j$. All logarithms in this paper are to base 2.

e_1 to be maximally certain effects.³ Note that our argument is not specific to the outcome combination $(0_f, 0_g)$ and $(1_f, 1_g)$ used in the the fine-grained uncertainty relation and choosing the remaining two inequalities corresponding to outcomes $(0_f, 1_g)$ and $(1_f, 0_g)$ leads to an analogous argument.

Example For our quantum example given by the states (5) we obtain

$$W \leq NkT \ln 2 \left(1 - \frac{1}{2} H(\zeta_{(0_x, 0_z)}) - \frac{1}{2} H(\zeta_{(1_x, 1_z)}) \right).$$

Equality is attained by taking $\{e_0, e_1\}$ to be the maximally certain effects given by the two eigenstates of $(X + Z)/\sqrt{2}$.

Second part of the cycle

In the second part we form a cycle (i.e., we go from ③ to ① in Figure 1)). We start with the completely mixed state ρ . Denote the different pure components of ρ by $\{q_j, \sigma_j\}_j$, i.e., $\rho = \sum_j q_j \sigma_j$. We can now ‘decompose’ ρ into its components by inserting a semi-transparent membrane which is opaque for a specific component σ_j , but completely transparent for all other components. Effectively, this membrane measures using the effects h_{σ_j} that are dual to the states σ_j . This membrane is used to confine all states σ_j in a volume $q_j V$. This is done for all components and we end up with a box where each component of ρ is sorted in a volume proportional to its weight in the convex combination. This process needs work proportional to $S(\rho)$.

In a second step, we create the (pure) components τ of $\rho_0 = \sum_j r_j^0 \tau_j^0$ and $\rho_1 = \sum_j r_j^1 \tau_j^1$ from the pure components of ρ and then ‘reassamble’ the states ρ_0 and ρ_1 . In order to do so, we subdivide the volumes containing σ_j into smaller volumes, such that the number of particles contained in these smaller volumes are proportional to $p_0 r_j^0$ and $p_1 r_j^1$. The pure state contained in each small volume is then transformed into the pure state τ_j^0 or τ_j^1 . Since these last states are also pure, no work is needed for this transformation. Finally, we ‘mix’ the different components of ρ_0 together, which allows us to extract work $p_0 S(\rho_0)$. Similarly we obtain work $p_1 S(\rho_1)$ from ρ_1 .

In total, the transformation $\rho \rightarrow \{p_i, \rho_i\}$, needs work

$$W = NkT \ln 2 (S(\rho) - \sum_i p_i S(\rho_i)).$$

Example Returning to the example above and using that the two eigenvalues of ρ are $1/2$, we obtain

$$S(\rho) = -2 \cdot \frac{1}{2} \log_2 \frac{1}{2} = 1.$$

Both ρ_0 and ρ_1 have the two eigenvalues $\{\frac{1}{2} + \frac{1}{2\sqrt{2}}, \frac{1}{2} - \frac{1}{2\sqrt{2}}\}$. Therefore,

$$S(\rho_i) = H\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right) \approx H(0.85).$$

The total work which has to be invested for this process is therefore given by

$$W = NkT \ln 2 \left(1 - H\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right) \right).$$

Closing the cycle

If we now perform the first and second process described above one after another (i.e., we perform a cycle, as depicted in Figure 1), the total work which can be extracted is given by

$$\Delta W = NkT \ln 2 \left(- \left(S(\rho) - \sum_i p_i S(\rho_i) \right) + \left(1 - \frac{1}{2} H(\zeta_{(f_0, g_0)}) - \frac{1}{2} H(\zeta_{(f_1, g_1)}) \right) \right).$$

In general, we can see that when the uncertainty relation is violated, this quantity can become positive and a positive ΔW corresponds to a violation of the second law of thermodynamics.

Example In our example, the above quantity corresponds to

$$\Delta W = NkT \ln 2 \left(H\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right) - \frac{1}{2} H(\zeta_{(0_x, 0_z)}) - \frac{1}{2} H(\zeta_{(1_x, 1_z)}) \right).$$

The fine-grained uncertainty relations for quantum mechanics state that $\zeta_{(0_x, 0_z)}, \zeta_{(1_x, 1_z)} \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}$. When this value is reached with equality, then $\Delta W = 0$ in the above calculation.

On the other hand if these values were larger, i.e., the uncertainty relation could be violated, then the binary entropy of them would be smaller and ΔW becomes positive.

DISCUSSION

We give a strong argument why quantum mechanical uncertainty relations should not be violated. Indeed, as

³ It is easy to see that in quantum mechanics the maximally certain effects e_0 and e_1 do indeed form a complete measurement in dimension 2.

we show, a violation of the uncertainty relations would lead to an ‘impossible machine’ which could extract net work from a cycle. Our result extends to more general theories than quantum theory - however, raises the question of which general form of entropy [19] is most significant. In quantum mechanics, the different entropies of [19] coincide, meaning that if a physical theory is just like quantum mechanics, but with a different amount of uncertainty, net work can be extracted.

Our cycle is similar to the ones given in [25–27], which study related questions: We can understand uncertainty relations as given in (1) as imposing a limit on how well one of several bits of information can be extracted from a qubit using the given measurements [12]. This means that the amount of uncertainty for all pairs of measurements that we could make directly imposes a limit on how much classical information we can store in each qubit. Indeed, in any system that is finite dimensional (possibly due to an energetic constraint), it is thus clear that the mere fact that we can only store a finite amount of information in a finite dimensional system (Holevo’s bound [28]) demands that non-commuting measurements obey uncertainty relations. This shows that our example is closely related to the ones given in [26, 27, 29, 30] where it has been shown that if it was possible to encode more than one bit of information in a qubit and therefore to violate the Holevo bound [28], then it was also possible to violate the second law of thermodynamics.

In [25] similar consequences had been shown if one was able to perfectly distinguish non-orthogonal quantum states. The possibility of distinguishing non-orthogonal states is again directly related to the question of how much information we can store in a quantum state.

In future work, it might be interesting to investigate whether an implication also holds in the other direction. Does any violation of the second law lead to a violation of the uncertainty relations?

We have investigated the relation between uncertainty and the second law of thermodynamics. A concept related to uncertainty is the one of complementarity. It is an open question, whether a violation of complementarity could also be used to build such an impossible machine.

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GENERAL THEORIES

In this appendix, we extend our result to more general physical theories. To this end, we need to introduce several additional assumptions and entropies. As quantum mechanics satisfies all assumptions made here, the derivation below can also be taken as a detailed explanation of the results claimed in the front matter.

In the main text, we have shown that a violation of uncertainty relations leads to a violation of the second law of thermodynamics for the quantum case. More precisely, we have assumed that all processes can be described by the quantum formalism, with the exception of the uncertainty relations. We now want to show that our result still holds when the physical processes have to be described by a general convex theory. We need several assumptions on these theories, which we clearly state below.

General assumptions

As already outlined we will make three very common assumptions on a generalized physical theory. The first two are thereby essentially made everywhere [17], the third is made very often (but not always (see e.g. [19])). We label assumptions as $A\cdot$. Whereas such assumptions may seem rather elaborate, there are physical reasons for assuming them. For example, a property known as bit symmetry [31] implies A3, A6 and A7.

A1: The state space Ω is convex.

A2: Effects are *linear* functionals.

A3: Pure states are dual to pure effects as outlined in the background section. Uncertainty relations can thus be stated equivalently in terms of states or measurements⁴.

Next, we will assume that pure effects are *projective* in that there exists a way to implement them in a physical measurement such that if we repeatedly apply an effect e_i by repeating the measurement we again obtain the same outcome. That is, $p(e_i e_i | \rho) = p(e_i | \rho)$ for all ρ ,

where with some abuse of notation we take $p(e_i e_i | \rho)$ to be the probability of observing e_i when making the measurement twice in a row, and don't consider the outcome of the first. A measurement is projective if it consists only of projective effects. Note that this is not the same as demanding that post-measurement states are the same for all ρ , which has significant consequences [32].

A4: Pure effects are projective.

We will also assume that the unit effect u , i.e. the effect satisfying $u(\rho) = 1$ for all ρ , has a dual state that is analogous to the maximally mixed state. If we accept duality between states and measurements, than this assumption is very natural.

A5: If $f_0 + f_1 = u$ for two effects f_0 and f_1 , then the dual states $\rho_{f_0} + \rho_{f_1} = \rho_u$ and $e(\rho_u/2) = 1/2$ for all pure effects e .

Note that this assumption again implies that we are dealing with the analogue of a qubit, i.e. a two-level system. It is possible to extend our statements for quantum mechanics to traceless two-outcome observables but as this requires additional assumptions in generality, we omit it. Our next assumption, however, is rather strong and significant, and extends beyond the duality of states and measurements. It is of course satisfied by quantum mechanics.

A6: Let $\rho = \sum_{j=1}^d q_j \sigma_j$ be a decomposition of ρ into perfectly distinguishable pure states σ_j : Let h_{σ_j} denote the pure effect dual to σ_j . Then $\sum_{j=1}^d h_{\sigma_j} = u$ and $h_k(\sigma_j) = \delta_{jk}$ for all j and k .

Finally, we will also need that pure states can be transformed into different pure states and that this does not require any work. In quantum mechanics, this is justified since the transformation just corresponds to a unitary.

A7: Let ρ and σ be pure states. Then the transformation $\rho \rightarrow \sigma$ is reversible (and thus does not require any work, neither can any work be gained from it).

Entropies

Several definitions of entropy are possible in generalized theories [19] that happen to coincide in quantum mechanics. The first is the so-called *decomposition entropy* given as

$$S(\rho) = \min_{\substack{\{p_j, \rho_j\}_j \\ \rho = \sum_{j=1}^d p_j \rho_j}} H(\{p_1, \dots, p_d\}), \quad (6)$$

where the minimization is taken over decompositions into pure states ρ_j and H is the Shannon entropy. Here, we will take the minimum over decompositions into perfectly

⁴ In [31] it was shown that such a duality holds at least for any theory which has a property called 'bit symmetry', which means that it allows for reversible computation.

distinguishable pure states. Note that the resulting quantity is equally well defined, but avoids an unnecessary strengthening of assumption A6. To define the other notion of entropy, we will need the following definition of *maximally fine-grained measurements* [19] as measurements such that each of its effects cannot be re-expressed as a non-trivial linear combination, i.e.,

$$\mathbf{e} = \{e_i\}_i : \text{maximally fine-grained} \Leftrightarrow \\ \text{for all } e_i : e_i = \alpha e'_\alpha + \beta e'_\beta, \alpha, \beta > 0 \Rightarrow e'_\alpha = e'_\beta.$$

We also call any effect satisfying this equation fine-grained. Note that pure effects are automatically maximally fine-grained. The *measurement entropy* is then given by

$$H(\rho) = \min_{\{e_j\}_{j=1}^\ell} H(\{e_1(\rho), \dots, e_\ell(\rho)\}), \quad (7)$$

where the minimization is taken over maximally fine-grained measurements. Finally, it would be possible to define entropies by a thermodynamical process itself [33], even in general physical theories [34]. In such a setting, also a difference between the decomposition and measurement entropy can lead to a violation of the second law [35]. As such, it is still under investigation what is the most relevant entropy [19] in general theories, also when it comes to operational tasks from quantum information such as decoupling [36, 37].

First part of the cycle

Below we state the explicit calculation of the work which can be extracted from the first part of the cycle. The measurements $\mathbf{f} = \{f_0, f_1\}$ and $\mathbf{g} = \{g_0, g_1\}$ are now not necessarily quantum mechanical, but do obey the assumptions above. We use a generalized notion of the completely mixed state and a projective measurement. In order to determine the position of the semi-transparent membranes in equilibrium, we assume that they perform a projective measurement, in the sense that a particle which is once measured to be e_0 (e_1) will give outcome e_0 (e_1) with certainty when the measurement is repeated, and never outcome e_1 (e_0). Additionally, we will use the following definitions in our calculation.

D1: ρ_0 is the mixture of ρ_{f_0} and ρ_{g_0} , and ρ_1 of ρ_{f_1} and ρ_{g_1} , i.e.,

$$\rho_0 = \frac{1}{2}(\rho_{f_0} + \rho_{g_0}) \\ \rho_1 = \frac{1}{2}(\rho_{f_1} + \rho_{g_1}).$$

D2: We choose an equal mixture of ρ_0 and ρ_1 , i.e., $p_i = 1/2$ for all i .

Note that by assumption A5 the state $\rho = \rho_0/2 + \rho_1/2$ has analogous properties to the completely mixed state in dimension 2, i.e., $p(e_j) = \sum_i p_i p(e_j|\rho_i) = 1/2$ for all j .

D3: We make a measurement with binary outcomes, i.e., $p(e_j|\rho_i) = 1 - p(e_{\bar{j}}|\rho_i)$.

The numbers on top of the equation refer to the assumptions and/or definitions stated above which are used in this step of the calculation.

$$\begin{aligned} W &= NkT \left(\sum_{i,j} p_i p(e_j|\rho_i) \ln(p_i p(e_j|\rho_i)) \right. \\ &\quad \left. - \sum_j p(e_j) \ln p(e_j) - \sum_i p_i \ln p_i \right) \\ &\stackrel{D2, A5}{=} NkT \ln 2 \left(\sum_{i,j} p_i p(e_j|\rho_i) \log(p_i p(e_j|\rho_i)) \right. \\ &\quad \left. - \log \frac{1}{2} - \log \frac{1}{2} \right) \\ &\stackrel{D2}{=} NkT \ln 2 \left(2 \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j} p(e_j|\rho_i) \left(\log \frac{1}{2} + \log p(e_j|\rho_i) \right) \right) \\ &= NkT \ln 2 \left(1 + \frac{1}{2} \sum_{i,j} p(e_j|\rho_i) (\log p(e_j|\rho_i)) \right) \\ &\stackrel{D3}{=} NkT \ln 2 \left(1 + \frac{1}{2} \sum_i (p(e_j|\rho_i) \log p(e_j|\rho_i) \right. \\ &\quad \left. + (1 - p(e_j|\rho_i)) \log(1 - p(e_j|\rho_i))) \right) \\ &= NkT \ln 2 \left(1 - \frac{1}{2} \sum_i H(p(e_j|\rho_i)) \right) \\ &= NkT \ln 2 \left(1 - \frac{1}{2} H \left(p \left(e_j \middle| \frac{1}{2} \rho_{f_0} + \frac{1}{2} \rho_{g_0} \right) \right) \right. \\ &\quad \left. - \frac{1}{2} H \left(p \left(e_j \middle| \frac{1}{2} \rho_{f_1} + \frac{1}{2} \rho_{g_1} \right) \right) \right) \\ &\stackrel{A2}{=} NkT \ln 2 \left(1 - \frac{1}{2} H \left(\frac{1}{2} p(e_j|\rho_{f_0}) + \frac{1}{2} p(e_j|\rho_{g_0}) \right) \right. \\ &\quad \left. - \frac{1}{2} H \left(p(e_j|\rho_{f_1}) + \frac{1}{2} p(e_j|\rho_{g_1}) \right) \right) \\ &\stackrel{A3}{=} NkT \ln 2 \left(1 - \frac{1}{2} H \left(\frac{1}{2} p(f_0|\rho_{e_j}) + \frac{1}{2} p(g_0|\rho_{e_j}) \right) \right. \\ &\quad \left. - \frac{1}{2} H \left(\frac{1}{2} p(f_1|\rho_{e_j}) + \frac{1}{2} p(g_1|\rho_{e_j}) \right) \right) \\ &\leq NkT \ln 2 \left(1 - \frac{1}{2} H(\zeta_{(f_0, g_0)}) - \frac{1}{2} H(\zeta_{(f_1, g_1)}) \right). \end{aligned} \quad (8)$$

Equality is achieved when the measurement can be formed from the maximally certain effects.

Second part of the cycle

We calculate the work needed for the second part of the cycle in two parts. First let us calculate the work needed to ‘decompose’ ρ into its different pure components. We use the effects h_{σ_j} , which are dual to the pure states σ_j which form the components of ρ , i.e., $\rho = \sum_j q_j \sigma_j$. Note that we can do this for any decomposition, in particular the one minimizing $S(\rho)$.

$$\begin{aligned}
W &= -NkT \ln 2 \left(\sum_j p(h_{\sigma_j} | \rho) \log p(h_{\sigma_j} | \rho) \right) \\
&= -NkT \ln 2 \left(\sum_j p \left(h_{\sigma_j} \middle| \sum_{j'} q_{j'} \sigma_{j'} \right) \right. \\
&\quad \left. \log p \left(h_{\sigma_j} \middle| \sum_{j'} q_{j'} \sigma_{j'} \right) \right) \\
&\stackrel{A2}{=} -NkT \ln 2 \left(\sum_j \left(\sum_{j'} q_{j'} p(h_{\sigma_j} | \sigma_{j'}) \right) \right. \\
&\quad \left. \log \left(\sum_{j'} q_{j'} p(h_{\sigma_j} | \sigma_{j'}) \right) \right) \\
&\stackrel{A6}{=} -NkT \ln 2 \left(\sum_{j'} q_{j'} \log q_{j'} \right)
\end{aligned}$$

$$= NkT \ln 2 S(\rho). \quad (9)$$

We then transform the pure states σ_j into the pure states τ_j^0 or τ_j^1 . By A7, this does not require any work. By performing a processes analogous to (9) but in the reverse direction, we can then extract work $NkT \ln 2 \sum_i p_i S(\rho_i)$ by ‘reassembling’ the states ρ_0 and ρ_1 . Overall, the work needed for the second part of the cycle is given by

$$W = NkT \ln 2 (S(\rho) - \sum_i p_i S(\rho_i)). \quad (10)$$

Closing the cycle

From the above calculation, i.e., by subtracting (10) from (8), we see that for the total cycle, the amount of work which can be extracted is given by

$$\begin{aligned}
\Delta W &= NkT \ln 2 \left(- \left(S(\rho) - \sum_i p_i S(\rho_i) \right) \right. \\
&\quad \left. + \left(1 - \frac{1}{2} H(\zeta_{(f_0, g_0)}) - \frac{1}{2} H(\zeta_{(f_1, g_1)}) \right) \right),
\end{aligned}$$

however, where S is now the general decomposition entropy. A net work gain of this cycle, and therefore a violation of the second law of thermodynamics, can therefore be reached if the uncertainty relations can be violated without at the same time changing the decomposition entropy.